

Solution of Quadratic Equation & Relation Between Roots & Co-Efficients

- (a) The solutions of the quadratic equation, $ax^2 + bx + c = 0$ is given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- (b) The expression $b^2 - 4ac = D$ is called the discriminant of the quadratic equation.
- (c) If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then;
- $\alpha + \beta = -b/a$
 - $\alpha\beta = c/a$
 - $|\alpha - \beta| = \sqrt{D}/|a|$
- (d) Quadratic equation whose roots are α & β is $(x - \alpha)(x - \beta) = 0$ i.e., $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ i.e. $x^2 - (\text{sum of roots})x + \text{product of roots} = 0$.

Nature of Roots

- (a) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ & $a \neq 0$ then;
- $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).
 - $D = 0 \Leftrightarrow$ roots are real & coincident (equal).
 - $D < 0 \Leftrightarrow$ roots are imaginary.
 - If $p + iq$ is one root of a quadratic equation, then the other root must be the conjugate $p - iq$ & vice versa.
($p, q \in R$ & $i = \sqrt{-1}$).
- (b) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in Q$ & $a \neq 0$ then;
- If D is a perfect square, then roots are rational.
 - If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & \sqrt{q} is a surd) then other root will be $p - \sqrt{q}$. (if a, b, c are rational). Because the coefficients are real

Common Roots of Two Quadratic Equations

- (a) Only one common root.

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then $a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha + c' = 0$.
By Cramer's Rule

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$$

$$\text{Therefore, } \alpha = \frac{ca' - c'a}{ab' - a'b} = \frac{bc' - b'c}{a'c - ac'}$$

So the condition for a common root is
 $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$

- (b) If both roots are same then $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

Roots Under Particular Cases

Let the quadratic equation $ax^2 + bx + c = 0$ has real roots and

- If $b = 0 \Rightarrow$ roots are of equal magnitude but of opposite sign
- If $c = 0 \Rightarrow$ one roots is zero other is $-b/a$
- If $a = c \Rightarrow$ roots are reciprocal to each other
- If $\begin{cases} a > 0, c < 0 \\ a < 0, c > 0 \end{cases} \Rightarrow$ roots are of opposite signs.
- If $\begin{cases} a > 0, b > 0, c > 0 \\ a < 0, b < 0, c < 0 \end{cases} \Rightarrow$ both roots are negative.
- If $\begin{cases} a > 0, b < 0, c > 0 \\ a < 0, b > 0, c < 0 \end{cases} \Rightarrow$ both roots are positive.
- If sign of $a =$ sign of $b \neq$ sign of $c \Rightarrow$ Greater root in magnitude is negative.
- If sign of $b =$ sign of $c \neq$ sign of $a \Rightarrow$ Greater root in magnitude is positive.
- If $a + b + c = 0 \Rightarrow$ one root is 1 and second root is c/a .

Maximum & Minimum Values of Quadratic Expression

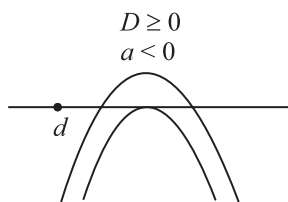
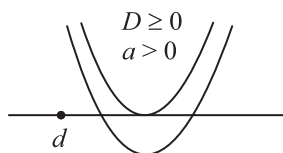
Maximum & Minimum Values of expression $y = ax^2 + bx + c$ is $\frac{-D}{4a}$ which occurs at $x = -(b/2a)$ according as $a < 0$ or $a > 0$.

$$y \in \left[\frac{-D}{4a}, \infty \right) \text{ if } a > 0 \quad \& \quad y \in \left(-\infty, \frac{-D}{4a} \right] \text{ if } a < 0.$$

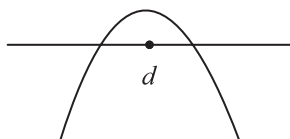
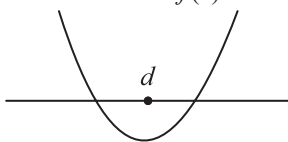
Location of Roots

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R, a \neq 0$

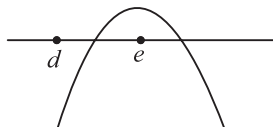
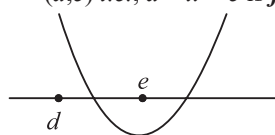
- (a) Conditions for both the roots of $f(x) = 0$ to be greater than a specified number ' d ' are $D \geq 0; a.f(d) > 0$ & $(-b/2a) > d$.



(b) Conditions for the both roots of $f(x) = 0$ to lie on either side of the number 'd' in other words the number 'd' lies between the roots of $f(x) = 0$ is $a \cdot f(d) < 0$.

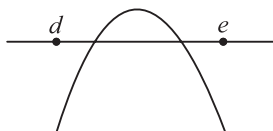
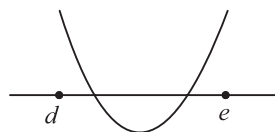


(c) Conditions for exactly one root of $f(x) = 0$ to lie in the interval (d, e) i.e., $d < x < e$ is $f(d) \cdot f(e) < 0$.



(d) Conditions that both roots of $f(x) = 0$ to be confined between the numbers d & e are (here $d < e$).

$$D \geq 0; a \cdot f(d) > 0 \text{ \& } a \cdot f(e) > 0; d < (-b/2a) < e$$



General Quadratic Expression in two Variables

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors if;

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \text{ OR } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Theory of Equations

If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are the roots of the equation; $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$ where a_0, a_1, \dots, a_n are constants $a_0 \neq 0$ then,

$$\sum \alpha_1 = -\frac{a_1}{a_0}, \sum \alpha_1 \alpha_2 = +\frac{a_2}{a_0}, \sum \alpha_1 \alpha_2 \alpha_3 = -\frac{a_3}{a_0}$$

Note:

(i) Every odd degree equation has at least one real root whose sign is opposite to that of its last term, when coefficient of highest degree term is (+)ve {If not then make it (+)ve}.

$$\text{Ex. } x^3 - x^2 + x - 1 = 0$$

(ii) Even degree polynomial whose last term is (-)ve & coefficient of highest degree term is (+)ve has atleast two real roots, one (+)ve & one (-)ve.

(iii) If equation contains only even power of x & all coefficient are (+)ve, then all roots are imaginary.